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Classification of eight-vertex solutions of the coloured Yang–Baxter equation

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Abstract. In this paper all eight-vertex type solutions of the coloured Yang–Baxter equation dependent on spectral as well as colour parameters are given. It is proved that they are composed of three groups of basic solutions, three groups of their degenerate forms and two groups of trivial solutions up to five solution transformations. Moreover, all non-trivial solutions can be classified into two types called Baxter type and free-fermion type.

0. Introduction

The Yang–Baxter or triangle equation which first appeared in [1–3] plays a prominent role in many branches of physics, for instance, in factorized S-matrices [4], exactly solvable models of statistical physics [5], complete integrable quantum and classical systems [6], quantum groups [7], conformal field theory and link invariants [8–11], to name just a few. In view of the importance of the Yang–Baxter equation, much attention has been directed to the search for solutions of the equation.

The coloured Yang–Baxter equation dependent on spectral as well as colour parameters is a generalization of the usual Yang–Baxter equation. It has also attracted a lot of research interest (see [12–16]) to find exact solutions for this type of Yang–Baxter equation. This is because the coloured Yang–Baxter equation concerns the free-fermion model in a magnetic field, multi-variable invariants of links and representations of quantum algebras and so on (see [17–21]).

The eight-vertex type solution of the coloured Yang–Baxter equation has been investigated previously in [12, 17, 20]. In [17] Fan and Wu first provided a single relation between the eight vertex weights in the general eight-vertex model by the pfaffian or dimer method, the so-called free-fermion condition. Based on this work, Bazhanov and Stroganov obtained an eight-vertex solution for the coloured Yang–Baxter equation in [12] devoted to the eight-vertex free-fermion model on a plane lattice. In [20] Murakami gave another eight-vertex solution in discussing multi-variable invariants of links. These are the only two eight-vertex solutions for the coloured Yang–Baxter equation we have known up to now.

The main theme of this paper is to give and classify all eight-vertex solutions of the coloured Yang–Baxter equation. The way to find the solution is from a computer algebra method given by Wu in [22]. Moreover, a theorem in [22] can prove that all solutions can be obtained by Wu's method. The paper is organized as follows. In section 1 we will

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review the coloured Yang–Baxter equation which in fact is a matrix equation. If the equation is expressed in component form it is composed of 28 polynomial equations in the eight-vertex case. We first introduce the symmetries or solution transformation for this system of equations and some concepts, including Hamiltonian coefficients, initial value condition, unitary condition and non-trivial gauge solution of the coloured Yang–Baxter equation. Using the symmetries we can simplify the system of equations as 12 polynomial equations. In section 2 we will apply the computer algebra method to 12 polynomial equations to get the algebraic curves and differential equations satisfied by the eight-vertex-type solution of the coloured Yang–Baxter equation. In this section we will also give two relations satisfied by the Hamiltonian coefficient which will play an important role in the classification of eight-vertex solutions of the coloured Yang–Baxter equation. Based on section 2, in section 3 we will construct all non-trivial gauge eight-vertex-type solutions of the coloured Yang–Baxter equation and classify them into two types called Baxter and free-fermion type. Section 4 is devoted to general solutions and the relation between Hamiltonian coefficients and spin-chain Hamiltonian. In sections 3 and 4 we will also show that the two solutions which appeared in [12] and [20] are special cases of the general solutions obtained in this paper.

In this paper a symbolic computation will be applied to accomplish some tedious computations and results obtained by computer calculations will be denoted by the symbol *.

1. The coloured Yang–Baxter equation, its symmetry and initial condition

By coloured Yang–Baxter equation we mean the following matrix equation

$$\begin{aligned} &\check{R}_{12}(u,\xi,\eta)\check{R}_{23}(u+v,\xi,\lambda)\check{R}_{12}(v,\eta,\lambda) = \check{R}_{23}(v,\eta,\lambda)\check{R}_{12}(u+v,\xi,\lambda)\check{R}_{23}(u,\xi,\eta) \\ &\check{R}_{12}(u,\xi,\eta) = \check{R}(u,\xi,\eta)\otimes E \qquad \check{R}_{23}(u,\xi,\eta) = E\otimes\check{R}(u,\xi,\eta) \end{aligned} \tag{1.1}$$

where $\check{R}(u,\xi,\eta)$ is a matrix function of N^2 dimension of u,ξ and η, E is the unit matrix of order N and $\otimes$ means the tensor product of two matrices. u, v are called spectral parameters and $\xi,\ \eta$ coloured parameters. If the matrix is independent of coloured parameters, then the coloured Yang–Baxter equation (1.1) will become the usual Yang–Baxter equation. If it is independent of spectral parameters then (1.1) will be reduced to the pure coloured Yang–Baxter equation:

$$\check{R}_{12}(\xi,\eta)\check{R}_{23}(\xi,\lambda)\check{R}_{12}(\eta,\lambda) = \check{R}_{23}(\eta,\lambda)\check{R}_{12}(\xi,\lambda)\check{R}_{23}(\xi,\eta). \tag{1.2}$$

For the coloured Yang–Baxter equation (1.1), the main interest in the paper is to discuss the solutions with the following form:

$$\check{R}(u,\xi,\eta) = \begin{pmatrix} R^{11}_{11}(u,\xi,\eta) & 0 & 0 & R^{11}_{22}(u,\xi,\eta) \\ 0 & R^{12}_{12}(u,\xi,\eta) & R^{12}_{21}(u,\xi,\eta) & 0 \\ 0 & R^{21}_{12}(u,\xi,\eta) & R^{21}_{21}(u,\xi,\eta) & 0 \\ R^{22}_{11}(u,\xi,\eta) & 0 & 0 & R^{22}_{22}(u,\xi,\eta) \end{pmatrix}. \tag{1.3}$$

The eight weight functions in (1.3) are denoted by

$$\begin{aligned} a_1(u,\xi,\eta) &= R^{11}_{11}(u,\xi,\eta) & a_5(u,\xi,\eta) &= R^{12}_{21}(u,\xi,\eta) \\ a_2(u,\xi,\eta) &= R^{12}_{12}(u,\xi,\eta) & a_6(u,\xi,\eta) &= R^{21}_{12}(u,\xi,\eta) \\ a_3(u,\xi,\eta) &= R^{21}_{21}(u,\xi,\eta) & a_7(u,\xi,\eta) &= R^{11}_{22}(u,\xi,\eta) \\ a_4(u,\xi,\eta) &= R^{22}_{22}(u,\xi,\eta) & a_8(u,\xi,\eta) &= R^{22}_{11}(u,\xi,\eta). \end{aligned}$$

We call solution (1.3) the eight-vertex-type solution if it satisfies in addition $a_i(u,\xi,\eta)\not\equiv 0$ $(i = 1, 2, \ldots, 8)$. Further, we only consider the solutions which are meromorphic functions of u, ξ and η. For notational simplicity, throughout this paper let

$$u_i = a_i(u, \xi, \eta) \qquad v_i = a_i(v, \eta, \lambda) \qquad w_i = a_i(u + v, \xi, \lambda) \qquad i = 1, 2, \ldots, 8.$$

For eight-vertex-type solutions, the matrix equation (1.1) is equivalent to the following 28 equations:

$$\left.\begin{aligned} u_7w_3v_8 - u_8w_2v_7 &= 0 \\ u_7w_8v_3 - u_8w_7v_2 &= 0 \\ u_2w_3v_2 - u_3w_2v_3 &= 0 \\ u_2w_8v_7 - u_3w_7v_8 &= 0 \end{aligned}\right\} \tag{1.4a}$$

$$\left.\begin{aligned} u_1w_5v_2 + u_7w_8v_6 - v_2w_1u_5 - v_5w_2u_3 &= 0 \\ u_1w_1v_7 + u_7w_3v_4 - v_7w_5u_5 - v_1w_7u_3 &= 0 \\ u_2w_6v_1 + u_5w_7v_8 - v_6w_1u_2 - v_3w_2u_6 &= 0 \\ u_1w_2v_1 + u_7w_4v_8 - v_2w_1u_2 - v_5w_2u_6 &= 0 \\ u_1w_7v_5 + u_7w_6v_3 - v_7w_5u_2 - v_1w_7u_6 &= 0 \\ u_1w_7v_2 + u_7w_6v_6 - v_1w_1u_7 - v_7w_2u_4 &= 0 \end{aligned}\right\} \tag{1.4b}$$

$$\left.\begin{aligned} u_4w_6v_2 + u_7w_8v_5 - v_2w_4u_6 - v_6w_2u_3 &= 0 \\ u_4w_4v_7 + u_7w_3v_1 - v_7w_6u_6 - v_4w_7u_3 &= 0 \\ u_2w_5v_4 + u_6w_7v_8 - v_5w_4u_2 - v_3w_2u_5 &= 0 \\ u_4w_2v_4 + u_7w_1v_8 - v_2w_4u_2 - v_6w_2u_5 &= 0 \\ u_4w_7v_6 + u_7w_5v_3 - v_7w_6u_2 - v_4w_7u_5 &= 0 \\ u_4w_7v_2 + u_7w_5v_5 - v_4w_4u_7 - v_7w_2u_1 &= 0 \end{aligned}\right\} \tag{1.4c}$$

$$\left.\begin{aligned} u_1w_5v_3 + u_8w_7v_6 - v_3w_1u_5 - v_5w_3u_2 &= 0 \\ u_1w_1v_8 + u_8w_2v_4 - v_8w_5u_5 - v_1w_8u_2 &= 0 \\ u_3w_6v_1 + u_5w_8v_7 - v_6w_1u_3 - v_2w_3u_6 &= 0 \\ u_1w_3v_1 + u_8w_4v_7 - v_3w_1u_3 - v_5w_3u_6 &= 0 \\ u_1w_8v_5 + u_8w_6v_2 - v_8w_5u_3 - v_1w_8u_6 &= 0 \\ u_1w_8v_3 + u_8w_6v_6 - v_1w_1u_8 - v_8w_3u_4 &= 0 \end{aligned}\right\} \tag{1.4d}$$

$$\left.\begin{aligned} u_4w_6v_3 + u_8w_7v_5 - v_3w_4u_6 - v_6w_3u_2 &= 0 \\ u_4w_4v_8 + u_8w_2v_1 - v_8w_6u_6 - v_4w_8u_2 &= 0 \\ u_3w_5v_4 + u_6w_8v_7 - v_5w_4u_3 - v_2w_3u_5 &= 0 \\ u_4w_3v_4 + u_8w_1v_7 - v_3w_4u_3 - v_6w_3u_5 &= 0 \\ u_4w_8v_6 + u_8w_5v_2 - v_8w_6u_3 - v_4w_8u_5 &= 0 \\ u_4w_8v_3 + u_8w_5v_5 - v_4w_4u_8 - v_8w_3u_1 &= 0. \end{aligned}\right\} \tag{1.4e}$$

Assume $\check{R}(u, \xi, \eta)$ is a solution of (1.1). Having carefully studied the system of equations (1.4), we find there are five symmetries for eight-vertex-type solutions of the coloured Yang–Baxter equation (1.1).

(A) Symmetry of interchanging indices. The system of equations (1.4) is invariant if we interchange the two sub-indices 2 and 3 as well as the two sub-indices 7 and 8 or the sub-indices 1 and 4 as well as the two sub-indices 5 and 6.

(B) The scaling symmetry. Multiplication of the solution $\check{R}(u, \xi, \eta)$ by an arbitrary function $g(u, \xi, \eta)$ is still a solution of the coloured Yang–Baxter equation (1.1).

(C) Symmetry of weight functions. If the weight functions $a_2(u, \xi, \eta)$, $a_3(u, \xi, \eta)$, $a_7(u, \xi, \eta)$ and $a_8(u, \xi, \eta)$ are replaced by the new weight functions

$$\bar{a}_2(u, \xi, \eta) = \frac{N(\xi)}{N(\eta)} a_2(u, \xi, \eta) \qquad \bar{a}_3(u, \xi, \eta) = \frac{N(\eta)}{N(\xi)} a_3(u, \xi, \eta)$$

$$\bar{a}_7(u, \xi, \eta) = sN(\eta)N(\xi) a_7(u, \xi, \eta) \qquad \bar{a}_8(u, \xi, \eta) = \frac{1}{sN(\eta)N(\xi)} a_8(u, \xi, \eta)$$

respectively, or $a_5(u,\xi,\eta)$ and $a_6(u,\xi,\eta)$ by $-a_5(u,\xi,\eta)$ and $-a_6(u,\xi,\eta)$, where $N(\xi)$ is an arbitrary function of coloured parameter and s is a complex constant, the new matrix $\check{R}(u,\xi,\eta)$ is still a solution of (1.1).

(D) Symmetry of spectral parameter. If we take a new spectral parameter $\bar{u}=\mu u$ where μ is a complex constant independent of spectral and coloured parameters, the new matrix $\check{R}(u',\xi,\eta)$ is still a solution of (1.1).

(E) Symmetry of colour parameters. If we take new coloured parameters $\zeta=f(\xi)$, $\theta=f(\eta)$, where $f(\xi)$ is an arbitrary function, then the new matrix $\check{R}(u,\zeta,\theta)$ is also a solution of (1.1).

The five symmetries (A)–(E) are called solution transformations $\boldsymbol{A}$–$\boldsymbol{E}$ of eight-vertex-type solutions of the coloured Yang–Baxter equation (1.1), respectively.

Dividing both sides of the third equation of (1.4*a*) by $a_2(u,\xi,\eta)a_2(u+v,\xi,\lambda)a_2(v,\eta,\lambda)$, we get

$$f(u+v,\xi,\lambda)=f(u,\xi,\eta)f(v,\eta,\lambda) \tag{1.5}$$

where $f(u,\xi,\eta)=a_3(u,\xi,\eta)/a_2(u,\xi,\eta)$. Putting $u=v=\eta=0$ in (1.5) we have

$$f(0,\xi,\lambda)=f(0,\xi,0)f(0,0,\lambda).$$

Substituting this formula into the one obtained by taking $u=v=\xi=0$ in (1.5) we get

$$f(0,0,\lambda)=f(0,0,\eta)f(0,\eta,\lambda)=f(0,0,\eta)f(0,\eta,0)f(0,0,\lambda).$$

This means

$$f(0,\eta,0)f(0,0,\eta)=1.$$

Otherwise, it is easy to show that $f(u,\xi,\eta)=0$, i.e. $a_3(u,\xi,\eta)=0$. Therefore

$$f(0,\xi,\eta)=\frac{M(\xi)}{M(\eta)} \tag{1.6}$$

where $M(\xi)=f(0,\xi,0)$. On the other hand, if we differentiate both sides of (1.5) with respect to the spectral variable v and then set $v=0$, $\lambda=\eta$, then

$$f'(u,\xi,\eta)=f'(0,\eta,\eta)f(u,\xi,\eta) \tag{1.7a}$$

holds, where the dot means derivative with respect to u and the simple formula

$$\left.\frac{\mathrm{d}G(u+v)}{\mathrm{d}v}\right|_{v=0}=\frac{\mathrm{d}G(u)}{\mathrm{d}u}$$

for any function $G(u)$, is used. Similarly, one also has

$$f'(v,\xi,\lambda)=f'(0,\xi,\xi)f(v,\xi,\lambda) \tag{1.7b}$$

if we differentiate both sides of (1.5) with respect to u and then set $u=0$ and $\eta=\xi$. The two formulae above imply $f'(0,\xi,\xi)$ is a constant independent of coloured parameter ξ. Hence

$$f(u,\xi,\eta)=\frac{M(\xi)}{M(\eta)}\exp(\nu u) \tag{1.8}$$

where ν is a complex constant.

From the group of equations (1.4*a*) we have

$$f(u,\xi,\eta)h(v,\eta,\lambda)=h(u+v,\xi,\lambda) \tag{1.9a}$$

$$f(u,\xi,\eta)f(v,\eta,\lambda)=f(u+v,\xi,\lambda) \tag{1.9b}$$

$$h(u,\xi,\eta)=h(u+v,\xi,\lambda)f(v,\eta,\lambda) \tag{1.9c}$$

$$h(u,\xi,\eta)=f(u+v,\xi,\lambda)h(v,\eta,\lambda) \tag{1.9d}$$

where $f(u,\xi,\eta) = a_3(u,\xi,\eta)/a_2(u,\xi,\eta)$, $h(u,\xi,\eta) = a_8(u,\xi,\eta)/a_7(u,\xi,\eta)$. If we let $v = 0$ and $\lambda = 0$ in (1.9d) then

$$h(u,\xi,\eta) = f(u,\xi,0)h(0,\eta,0).$$

Substituting this into (1.9c) and using (1.9b), one obtains

$$f(u,\xi,0)h(0,\eta,0) = f(u,\xi,0)f(v,0,0)h(0,\lambda,0)f(v,\eta,\lambda)$$

or

$$h(0,\eta,0) = f(v,0,0)h(0,\lambda,0)f(v,\eta,\lambda).$$

This formula implies $\nu = 0$ in (1.8) and then one can obtain

$$f(u,\xi,\eta) = \frac{M(\xi)}{M(\eta)} \qquad h(u,\xi,\eta) = lM(\xi)M(\eta)$$

where l is a complex constant independent of spectral and coloured parameters.

So, up to the solution transformation $\boldsymbol{B}$ and $\boldsymbol{C}$ one can assume

$$a_3(u,\xi,\eta) = a_2(u,\xi,\eta) = 1 \qquad a_8(u,\xi,\eta) = a_7(u,\xi,\eta)$$

without losing generality. The case (1.4a)–(1.4d) can be simplified to the following 12 equations,

$$\begin{aligned}
&a_5(v,\eta,\lambda) + a_5(u,\xi,\eta)a_1(u+v,\xi,\lambda) - a_1(u,\xi,\eta)a_5(u+v,\xi,\lambda)\\
&\qquad -a_7(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_6(v,\eta,\lambda) = 0\\
&a_7(u+v,\xi,\lambda)a_1(v,\eta,\lambda) + a_5(u,\xi,\eta)a_5(u+v,\xi,\lambda)a_7(v,\eta,\lambda)\\
&\qquad -a_1(u,\xi,\eta)a_1(u+v,\xi,\lambda)a_7(v,\eta,\lambda) - a_7(u,\xi,\eta)a_4(v,\eta,\lambda) = 0\\
&a_6(u,\xi,\eta) + a_1(u+v,\xi,\lambda)a_6(v,\eta,\lambda) - a_6(u+v,\xi,\lambda)a_1(v,\eta,\lambda)\\
&\qquad -a_5(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_7(v,\eta,\lambda) = 0\\
&a_6(u,\xi,\eta)a_5(v,\eta,\lambda) + a_1(u+v,\xi,\lambda) - a_1(u,\xi,\eta)a_1(v,\eta,\lambda)\\
&\qquad -a_7(u,\xi,\eta)a_4(u+v,\xi,\lambda)a_7(v,\eta,\lambda) = 0\\
&a_6(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_1(v,\eta,\lambda) + a_5(u+v,\xi,\lambda)a_7(v,\eta,\lambda)\\
&\qquad -a_1(u,\xi,\eta)a_7(u+v,\xi,\lambda)a_5(v,\eta,\lambda) - a_7(u,\xi,\eta)a_6(u+v,\xi,\lambda) = 0\\
&a_7(u,\xi,\eta)a_1(u+v,\xi,\lambda)a_1(v,\eta,\lambda) + a_4(u,\xi,\eta)a_7(v,\eta,\lambda) - a_1(u,\xi,\eta)a_7(u+v,\xi,\lambda)\\
&\qquad -a_7(u,\xi,\eta)a_6(u+v,\xi,\lambda)a_6(v,\eta,\lambda) = 0
\end{aligned} \tag{1.10}$$

plus six equations obtained by interchanging the sub-indices 1 and 4 as well as 5 and 6 in each of equations (1.10). We call the six equations *counterparts* of (1.10).

Now we solve the equations obtained by letting $u = 0$ and $\eta = \xi$ in (1.10) with respect to the variables $\{a_1(0,\xi,\xi),\ a_4(0,\xi,\xi),\ a_5(0,\xi,\xi),\ a_6(0,\xi,\xi),\ a_7(0,\xi,\xi)\}$. It is easy to prove the following.

Proposition 1.1. For a solution of equations (1.10), weight functions satisfy

$$\begin{aligned}
&a_1(0,\xi,\xi) = a_4(0,\xi,\xi) = 1\\
&a_5(0,\xi,\xi) = a_6(0,\xi,\xi) = a_7(0,\xi,\xi) = a_8(0,\xi,\xi) = 0.
\end{aligned} \tag{1.11}$$

Otherwise, up to the solution transformations $\boldsymbol{A}$–$\boldsymbol{E}$, we have two trivial solutions of the coloured Yang–Baxter equation (1.1). The first is

$$\begin{aligned}
&a_1(u,\xi,\eta) = a_4(u,\xi,\eta) = a_5(u,\xi,\eta) = a_6(u,\xi,\eta) = H(u,\xi,\eta)\\
&a_2(u,\xi,\eta) = a_3(u,\xi,\eta) = a_7(u,\xi,\eta) = a_8(u,\xi,\eta) = 1
\end{aligned} \tag{1.12a}$$

where $H(u,\xi,\eta)$ is an arbitrary function of spectral parameter u and coloured parameters ξ,η and the second

$$a_1(u,\xi,\eta)=a_4(u,\xi,\eta)=a_5(u,\xi,\eta)=-a_6(u,\xi,\eta)=\frac{F(\xi)}{F(\eta)}\exp(u)$$
$$a_1(u,\xi,\eta)=a_2(u,\xi,\eta)=1 \qquad a_7(u,\xi,\eta)=a_8(u,\xi,\eta)=\mathrm{i} \tag{1.12b}$$

where $\mathrm{i}^2=-1$ and $F(\xi)$ is an arbitrary function of coloured parameter ξ.

Definition 1.2. By a gauge solution of the coloured Yang–Baxter equation (1.1) we mean the solution whose weight functions satisfy $a_2(u,\xi,\eta)=a_3(u,\xi,\eta)=1$ and $a_7(u,\xi,\eta)=a_8(u,\xi,\eta)$ and the condition (1.11).

(1.11) is called an initial condition of gauge solutions. The condition is simple but very important. It will be quoted again and again in finding gauge solutions of the coloured Yang–Baxter equation. For example, taking $v=-u$, $\lambda=\xi$ in (1.10) and their counterparts and using the initial condition (1.11), one has the following equations:

$$\begin{aligned}
&a_6(-u,\eta,\xi)=-a_6(u,\xi,\eta)\\
&a_5(-u,\eta,\xi)=-a_5(u,\xi,\eta)\\
&1-a_1(u,\xi,\eta)a_1(-u,\eta,\xi)+a_6(u,\xi,\eta)a_5(-u,\eta,\xi)-a_7(u,\xi,\eta)a_7(-u,\eta,\xi)=0\\
&1-a_4(u,\xi,\eta)a_4(-u,\eta,\xi)+a_5(u,\xi,\eta)a_6(-u,\eta,\xi)-a_7(u,\xi,\eta)a_7(-u,\eta,\xi)=0.
\end{aligned}\tag{1.13}$$

The unitary condition of a solution of the Yang–Baxter equation means

$$\check{R}(0,\xi,\xi)=E \qquad \check{R}(u,\xi,\eta)\check{R}(-u,\eta,\xi)=g(u,\xi,\eta)E$$

where E is the unit matrix and $g(u,\xi,\eta)$ a scalar function. Hence, it is easy to get from (1.13) that

Proposition 1.3. For gauge solutions $\check{R}(u,\xi,\eta)$ of the coloured Yang–Baxter equation (1.1), the unitary condition is

$$\check{R}(u,\xi,\eta)\check{R}(-u,\eta,\xi)=(1-a_5(u,\xi,\eta)a_6(u,\xi,\eta))E.$$

Differentiating both sides of all equations in (1.10) and their counterparts with respect to the variable v and letting $v=0$, $\lambda=\eta$, by virtue of the initial condition (1.11) one immediately obtains

$$\begin{aligned}
&m_5(\eta)+a_5(u,\xi,\eta)a_1'(u,\xi,\eta)-a_1(u,\xi,\eta)a_5'(u,\xi,\eta)-m_6(\eta)a_7(u,\xi,\eta)^2=0\\
&a_7'(u,\xi,\eta)+(m_1(\eta)-m_4(\eta))a_7(u,\xi,\eta)+m_7(\eta)(a_5(u,\xi,\eta)^2-a_1(u,\xi,\eta)^2)=0\\
&a_6'(u,\xi,\eta)-m_6(\eta)a_1(u,\xi,\eta)+m_1(\eta)a_6(u,\xi,\eta)+m_7(\eta)a_5(u,\xi,\eta)a_7(u,\xi,\eta)=0\\
&a_1'(u,\xi,\eta)-m_1(\eta)a_1(u,\xi,\eta)+m_5(\eta)a_6(u,\xi,\eta)-m_7(\eta)a_4(u,\xi,\eta)a_7(u,\xi,\eta)=0\\
&a_6(u,\xi,\eta)a_7'(u,\xi,\eta)-a_7(u,\xi,\eta)a_6'(u,\xi,\eta)+m_1(\eta)a_6(u,\xi,\eta)a_7(u,\xi,\eta)\\
&\qquad -m_5(\eta)a_1(u,\xi,\eta)a_7(u,\xi,\eta)+m_7(\eta)a_5(u,\xi,\eta)=0\\
&a_7(u,\xi,\eta)a_1'(u,\xi,\eta)-a_1(u,\xi,\eta)a_7'(u,\xi,\eta)+m_1(\eta)a_1(u,\xi,\eta)a_7(u,\xi,\eta)\\
&\qquad -m_6(\eta)a_6(u,\xi,\eta)a_7(u,\xi,\eta)+m_7(\eta)a_4(u,\xi,\eta)=0
\end{aligned}\tag{1.14a}$$

and their counterparts, where and throughout the paper we denote

$$a_i'(u,\xi,\eta)=\frac{\partial}{\partial u}a_i(u,\xi,\eta) \qquad m_i(\xi)=a_i'(u,\xi,\eta)|_{\{u=0,\eta=\xi\}}$$

for $i=1$, 4–7.

We call $m_i(\xi)$ Hamiltonian coefficients of weight functions with respect to spectral parameter or simply coefficients. Sometimes we write m_i instead of $m_i(\xi)$ for brevity.

If we differentiate (1.10) with respect to u and let $u = 0$, $\eta = \xi$ and then replace the variables v and λ by u and η, then we have

$$
\begin{aligned}
&m_6(\xi) + a_6(u,\xi,\eta)a_1'(u,\xi,\eta) - a_1(u,\xi,\eta)a_6'(u,\xi,\eta) - m_5(\xi)a_7(u,\xi,\eta)^2 = 0\\
&a_7'(u,\xi,\eta) + (m_1(\xi) - m_4(\xi))a_7(u,\xi,\eta) + m_7(\xi)(a_6(u,\xi,\eta)^2 - a_1(u,\xi,\eta)^2) = 0\\
&a_5'(u,\xi,\eta) - m_5(\xi)a_1(u,\xi,\eta) + m_1(\xi)a_5(u,\xi,\eta) + m_7(\xi)a_6(u,\xi,\eta)a_7(u,\xi,\eta) = 0\\
&a_1'(u,\xi,\eta) - m_1(\xi)a_1(u,\xi,\eta) + m_6(\xi)a_5(u,\xi,\eta) - m_7(\xi)a_4(u,\xi,\eta)a_7(u,\xi,\eta) = 0\\
&a_5(u,\xi,\eta)a_7'(u,\xi,\eta) - a_7(u,\xi,\eta)a_5'(u,\xi,\eta) + m_1(\xi)a_5(u,\xi,\eta)a_7(u,\xi,\eta)\\
&\qquad -m_6(\xi)a_1(u,\xi,\eta)a_7(u,\xi,\eta) + m_7(\xi)a_6(u,\xi,\eta) = 0\\
&a_7(u,\xi,\eta)a_1'(u,\xi,\eta) - a_1(u,\xi,\eta)a_7'(u,\xi,\eta) + m_1(\xi)a_1(u,\xi,\eta)a_7(u,\xi,\eta)\\
&\qquad -m_5(\xi)a_5(u,\xi,\eta)a_7(u,\xi,\eta) + m_7(\xi)a_4(u,\xi,\eta) = 0
\end{aligned}
\tag{1.14b}
$$

and their counterparts.

Remark 1. In this paper, the following trick will often be employed to obtain the equations such as (1.14*a*) and (1.14*b*). We first do the calculation with respect to v and take $\lambda = \eta$ to yield some equations. Then we repeat the same operation with u in place of v and take $\eta = \xi$ to yield another equation. Then we compare the two results to reduce some of the formulae. For example, formula (1.8) is obtained by this method.

Comparing (1.14*a*) with (1.14*b*), we found that the trick, in fact, is to interchange the sub-indexes 5 and 6 (or 1 and 4) and then replace $m_i(\eta)$ by $m_i(\xi)$ in the original equations (1.14*a*). We call this trick *symmetric operation*.

Remark 2. If the variable with respect to which we differentiate (1.10) is not a spectral parameter v but a coloured parameter λ, the equations obtained are the same as (1.14*a*) if we still let $v = 0$ and $\lambda = \eta$. Of course, then the dot means the derivative with respect to the coloured parameter η, that is the second coloured variable in $a_i(u,\xi,\eta)$, and $m_i(\eta) = \mathrm{d}a_i(v,\eta,\lambda)/\mathrm{d}\lambda|_{v=0,\lambda=\eta}$. Similarly, (1.14*b*) also represents the equations obtained by differentiating (1.10) with respect to the first coloured variable ξ in $a_i(u,\xi,\eta)$, but in this case the dot means the derivative with respect to ξ and $m_i(\xi) = \mathrm{d}a_i(u,\xi,\eta)/\mathrm{d}\lambda|_{u=0,\eta=\xi}$.

2. The coefficients, curves and differential equations of weight functions

In this section we will discuss properties of Hamiltonian coefficients, curves and differential equations satisfied by weight functions.

It follows from the second equation of (1.14*a*) and its counterpart that

$$2a_7'(u,\xi,\eta) + m_7(\eta)(a_5(u,\xi,\eta)^2 + a_6(u,\xi,\eta)^2 - a_1(u,\xi,\eta)^2 - a_4(u,\xi,\eta)^2) = 0. \tag{2.1}$$

If the symmetric operation is used then one also has

$$2a_7'(u,\xi,\eta) + m_7(\xi)(a_5(u,\xi,\eta)^2 + a_6(u,\xi,\eta)^2 - a_1(u,\xi,\eta)^2 - a_4(u,\xi,\eta)^2) = 0. \tag{2.2}$$

We know $(u_5^2 + u_6^2 - u_1^2 - u_4^2) \not\equiv 0$ due to the initial condition (1.11). Compared the formulae (2.1) and (2.2), one shows

$$m_7(\xi) = m_7(\eta).$$

If $m_7(\xi) = 0$ then

$$a_7'(u,\xi,\eta) = 0 \qquad m_1(\eta) = m_4(\eta)$$

due to the second of (1.14a) and its counterpart. This implies $a_7(u, \xi, \eta)$ is a function of only the coloured variables ξ and η.

Proposition 2.1. For gauge eight-vertex solutions, $m_7(\xi)$ is a constant independent of coloured parameters and $a_7(u, \xi, \eta)$ is independent of spectral parameter if $m_7(\eta) = 0$.

Remark 3. In fact, as mentioned in remark 2, the latter property of proposition 2.1 also holds for coloured parameter, namely, $a_7(u, \xi, \eta)$ will be independent of the coloured parameter ξ (or η) if $\mathrm{d}/\mathrm{d}\xi(a_7(u, \xi, \eta))|_{u=0,\eta=\xi} = 0$ (or $\mathrm{d}/\mathrm{d}\eta(a_7(u, \xi, \eta))|_{u=0,\xi=\eta} = 0$).

In what follows we denote $m_7(\xi) = \alpha$.

Furthermore, if $\alpha = 0$, letting $u = 0$, $\eta = \xi$ in the following equation

$$a_1'(u, \xi, \eta) = m_6(\eta)a_6(u, \xi, \eta) - m_1(\eta)a_1(u, \xi, \eta) \tag{2.3}$$

which is from the sixth equation in (1.14a), we obtain $m_1(\eta) = 0$ owing to the initial condition (1.11). Therefore $m_4(\xi) = 0$. Similarly, we have

$$a_6'(u, \xi, \eta) = m_1(\eta)a_6(u, \xi, \eta) - m_5(\eta)a_1(u, \xi, \eta) \tag{2.4}$$

from the fifth equation in (1.14a) and then

$$m_6(\eta) = -m_5(\eta). \tag{2.5}$$

Substituting $m_6(\eta) = -m_5(\eta)$ and $m_1(\eta) = m_4(\eta) = 0$ into the third and fourth equations of (1.14a) and their counterparts, we have the following differential equations:

$$\begin{aligned} a_1'(u, \xi, \eta) &= -m_5(\eta)a_6(u, \xi, \eta) \qquad & a_4'(u, \xi, \eta) &= m_5(\eta)a_5(u, \xi, \eta) \\ a_6'(u, \xi, \eta) &= -m_5(\eta)a_1(u, \xi, \eta) \qquad & a_5'(u, \xi, \eta) &= m_5(\eta)a_4(u, \xi, \eta). \end{aligned} \tag{2.6}$$

Therefore, if $\alpha = 0$ then weight functions satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} a_i(u, \xi, \eta) = m_5(\eta)^2 a_i(u, \xi, \eta) \qquad i = 1, 4, 5, 6. \tag{2.7}$$

Furthermore, using the symmetric operation

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} a_i(u, \xi, \eta) = m_5(\xi)^2 a_i(u, \xi, \eta) \qquad i = 1, 4, 5, 6 \tag{2.8}$$

hold. Hence $m_5(\eta)$ actually is a constant independent of coloured parameters and not identically zero, otherwise, the solutions will be independent of spectral parameter. Thus we can let $m_5(\eta) = \beta$.

The argument above implies the following proposition.

Proposition 2.2. For a gauge eight-vertex solution, there exists at least one between $m_7(\xi)$ and $m_5(\xi)$ (or $m_6(\xi)$) which is not zero identically. Otherwise, the solution will be independent of spectral parameter.

As for the Hamiltonian coefficients $m_1(\xi)$ and $m_4(\xi)$ we have

*Proposition 2.3**. For gauge eight-vertex solutions of the coloured Yang–Baxter equation (1.1)

$$m_1(\xi)^2 - m_4(\xi)^2 = 0.$$

Proof. As first step, we regard the weight functions w_i ($i = 1$, 4–7) in (1.10) and their counterparts as indeterminates, The left-hand side of each of (1.10) and its counterpart are polynomial functions of the indeterminates. After eliminating the five indeterminates $\{w_1, w_4, w_5, w_6, w_7\}$ in numerically increasing order with respect to the system of equations (1.10) we can obtain seven equations which do not contain the indeterminates w_i ($i = 1$, 4–7). Then differentiating them with respect to the spectral variable v, letting $v = 0$, $\lambda = \eta$ and then substituting the initial values (1.11) into the resulting ones, we obtain the following seven polynomial equations:

$$\begin{aligned}
&m_7u_1^3 - m_7u_1u_5^2 - 3m_1u_1u_7 + m_4u_1u_7 - m_7u_4u_7^2 - m_7u_4 + m_5u_6u_7 + m_6u_6u_7 = 0\\
&-m_6u_1u_4 + m_7u_1u_6u_7 + m_7u_4u_5u_7 + m_1u_4u_6 + m_4u_4u_6 - m_6u_5u_6 - m_5u_7^2 + m_6 = 0\\
&m_7u_1^2 - m_7u_4^2 - m_7u_5^2 + m_7u_6^2 + 2m_4u_7 - 2m_1u_7 = 0\\
&-m_5u_1u_4 + m_1u_1u_5 + m_4u_1u_5 + m_7u_1u_6u_7 + m_7u_4u_5u_7 - m_5u_5u_6 - m_6u_7^2 + m_5 = 0\\
&m_7u_1^2u_6 - m_6u_1u_7 - m_5u_1u_7 - m_7u_5^2u_6 + m_7u_5u_7^2 + m_7u_5 + m_4u_6u_7 + m_1u_6u_7 = 0\\
&m_7u_1^3u_5 - m_6u_1u_4u_7 - m_7u_1u_5^3 + 2m_4u_1u_5u_7 - 2m_1u_1u_5u_7 + m_7u_1u_6 - m_7u_4u_5u_7^2\\
&\qquad +m_5u_5u_6u_7 + m_6u_7^3 - m_5u_7 = 0\\
&-m_7u_1^2u_4 + m_7u_1u_7^2 + m_7u_1 + m_7u_4u_5^2 + m_4u_4u_7 + m_1u_4u_7 - m_5u_5u_7 - m_6u_5u_7 = 0.
\end{aligned} \tag{2.9}$$

As a second step, we think of m_i as indeterminates and first eliminate m_1, m_4 and m_5 to get two systems of equations, which are equivalent to (2.9). The first is

$$\begin{aligned}
&-m_5u_1u_4 + m_1u_1u_5 + m_4u_1u_5 + m_7u_1u_6u_7 + m_7u_4u_5u_7 - m_5u_5u_6 - m_6u_7^2 + m_5 = 0\\
&-m_7u_1^3u_5 + m_7u_1u_4^2u_5 + 2m_5u_1u_4u_7 + m_7u_1u_5^3 - m_7u_1u_5u_6^2 - 4m_4u_1u_5u_7 - 2m_7u_1u_6u_7^2\\
&\qquad -2m_7u_4u_5u_7^2 + 2m_5u_5u_6u_7 + 2m_6u_7^3 - 2m_5u_7 = 0\\
&-m_7u_1u_4^2u_5 + m_6u_1u_4u_7 + m_7u_1u_5u_6^2 - m_7u_1u_6 + m_7u_4u_5u_7^2 - m_5u_5u_6u_7 - m_6u_7^3\\
&\qquad +m_5u_7 = 0.
\end{aligned} \tag{2.10}$$

which contain m_1, m_4 and m_5. The second is

$$\begin{aligned}
&(u_1u_4 + u_5u_6 - u_7^2 - 1)(-m_7u_1u_4^2u_5 + m_6u_1u_4u_7 + m_7u_1u_5u_6^2 - m_7u_1u_6 + m_7u_4u_5\\
&\qquad -m_6u_5u_6u_7) = 0\\
&u_1(u_1u_4 + u_5u_6 - u_7^2 - 1)(m_7u_1u_4u_5^2 - m_6u_1u_5u_7 - m_7u_4^2u_5u_6 + m_6u_4u_6u_7 - m_7u_5^3u_6\\
&\qquad +m_7u_5^2 + m_7u_5u_6^3 - m_7u_6^2) = 0\\
&u_1(u_1u_4 + u_5u_6 - u_7^2 - 1)(-m_7u_4^3u_5u_6 + m_6u_4^2u_6u_7 + m_7u_4u_5^2u_7^2 + m_7u_4u_5u_6^3 - m_7u_4u_6^2\\
&\qquad -m_6u_5^2u_6u_7 - m_6u_5u_7^3 + m_6u_5u_7) = 0\\
&u_1(u_1u_4 + u_5u_6 - u_7^2 - 1)(-m_7u_1u_5^2u_6 + m_7u_1u_5 - m_7u_4^3u_5 + m_6u_4^2u_7 + m_7u_4u_5^3\\
&\qquad +m_7u_4u_5u_6^2 - m_7u_4u_6 - m_6u_5^2u_7) = 0
\end{aligned} \tag{2.11}$$

which do not contain m_1, m_4 and m_5. So the free-fermion condition [17]

$$u_1u_4 + u_5u_6 - 1 - u_7^2 = 0 \tag{2.12}$$

or

$$\begin{aligned}
&-m_7u_1u_4^2u_5 + m_6u_1u_4u_7 + m_7u_1u_5u_6^2 - m_7u_1u_6 + m_7u_4u_5 - m_6u_5u_6u_7 = 0\\
&m_7u_1u_4u_5^2 - m_6u_1u_5u_7 - m_7u_4^2u_5u_6 + m_6u_4u_6u_7 - m_7u_5^3u_6 + m_7u_5^2 + m_7u_5u_6^3 - m_7u_6^2 = 0\\
&-m_7u_4^3u_5u_6 + m_6u_4^2u_6u_7 + m_7u_4u_5^2u_7^2 + m_7u_4u_5u_6^3 - m_7u_4u_6^2 - m_6u_5^2u_6u_7 - m_6u_5u_7^3\\
&\qquad +m_6u_5u_7 = 0
\end{aligned}$$

$$-m_7u_1u_5^2u_6+m_7u_1u_5-m_7u_4^3u_5+m_6u_4^2u_7+m_7u_4u_5^3+m_7u_4u_5u_6^2-m_7u_4u_6$$
$$-m_6u_5^2u_7=0 \tag{2.13}$$

will hold. In the third step, applying the fourth equation in (2.13) as a main equation to kill the indeterminate m_6 in the three equations remaining in (2.13) and then performing factorization of the new polynomial equations after killing m_6, one can obtain that

$$-m_7u_1u_5^2u_6+m_7u_1u_5-m_7u_4^3u_5+m_6u_4^2u_7+m_7u_4u_5^3+m_7u_4u_5u_6^2-m_7u_4u_6$$
$$-m_6u_5^2u_7=0$$
$$m_7(u_5u_6-1)(u_1^2u_5-2u_1u_4u_6+u_4^2u_5-u_5^3+u_5u_6^2)u_5=0$$
$$m_7(u_5u_6-1)(-u_1u_4^2u_6+u_1u_5^2u_6+u_1u_5u_7^2-u_1u_5+u_4^3u_5-u_4u_5^3-u_4u_6u_7^2$$
$$+u_4u_6)u_5=0$$
$$m_7(u_5u_6-1)(-u_1^2u_4+2u_1u_5u_6+u_4^3-u_4u_5^2-u_4u_6^2)u_5=0 \tag{2.14}$$

is equivalent to (2.13). It follows from the first equation of (2.14) that $m_7 \not\equiv 0$. Otherwise, thanks to proposition 2.2, $a_4(u,\xi,\eta)^2=a_5(u,\xi,\eta)^2$. The latter is impossible thanks to the initial condition (1.11). Hence the following three equations

$$u_1^2u_5-2u_1u_4u_6+u_4^2u_5-u_5^3+u_5u_6^2=0$$
$$-u_1u_4^2u_6+u_1u_5^2u_6+u_1u_5u_7^2-u_1u_5+u_4^3u_5-u_4u_5^3-u_4u_6u_7^2+u_4u_6=0$$
$$-u_1^2u_4+2u_1u_5u_6+u_4^3-u_4u_5^2-u_4u_6^2=0 \tag{2.15}$$

and the first of (2.14) is equivalent to (2.14), where we use the initial condition (1.11) again to yield $1-u_5u_6 \not\equiv 0$.

Finally, if we differentiate (2.12) and the third equation of (2.15) with respect to u and let $u=0$, $\xi=\eta$ and then apply the initial conditions (1.11) again , we come to the conclusion of proposition 2.3. □

Remark 4. When we perform the operation of eliminating indeterminates with respect to a system of polynomial equations, according to the theorem of zero structure of algebraic varieties [22], the coefficient of the term with the highest degree of the indeterminate in the main polynomial equation (to be eliminated in other polynomials) should not be identified with zero. in the event it is identified with zero, we should add the coefficient into the system of equations to produce a new system of equations. Otherwise, it is possible to lose some solutions. For example, when we use the fourth equation in (2.13) to eliminate m_6 in the three remaining ones in (2.13), because the coefficient of m_6 in the fourth equation of (2.13) is $u_7(u_4^2-u_5^2)$ which does not identify with zero due to the initial condition (1.11), (2.14) is equivalent to the system of equations (2.13).

From the argument proving proposition 2.3 above we see the system of equations (2.9) is equivalent to two groups of equations. The first is (2.10) and (2.15) plus the first of (2.14). The second is (2.10) and (2.12), the free-fermion condition.

Now we consider the two cases respectively. For the first case we differentiate the second equation in (2.15) and take $u=0$, $\xi=\eta$. Then we substitute the initial condition (1.11) into the result to get

$$m_5(\eta)=m_6(\eta). \tag{2.16}$$

By performing factorization of the equation obtained by eliminating u_4 in the third equation of (2.15) and by using the second equation of (2.15) , we can get

$$2u_6(u_6-u_5)(u_6+u_5)(u_1-u_5)(u_1+u_5)(u_1-u_6)(u_1+u_6)=0. \tag{2.17}$$

Together, (2.17) and the initial condition (1.11) imply

$$a_5(u,\xi,\eta) = a_6(u,\xi,\eta) \tag{2.18a}$$

or

$$a_5(u,\xi,\eta) = -a_6(u,\xi,\eta) \tag{2.18b}$$

hold. Substituting (2.18*a*) into the third equation in (2.15), we then see

$$a_1(u,\xi,\eta) = a_4(u,\xi,\eta). \tag{2.19}$$

Substituting (2.18*a*) and (2.19) into the first equation in (2.14), we have

$$(u_5 - u_1(u_5+u_1)(\alpha u_5 u_1 - m_6 u_7) = 0. \tag{2.20}$$

So

$$\alpha a_1(u,\xi,\eta)a_5(u,\xi,\eta) - m_6(\eta)a_7(u,\xi,\eta) = 0 \tag{2.21}$$

where the initial condition (1.11) is used again. (2.21) implies $m_6(\eta) \not\equiv 0$, or α will also identify with zero, but this will contradict proposition 2.2. If we perform the symmetric operation with respect to (2.21) and use (2.19), then

$$\alpha a_1(u,\xi,\eta)a_5(u,\xi,\eta) - m_6(\xi)a_7(u,\xi,\eta) = 0 \tag{2.22}$$

(2.21) and (2.22) will imply $m_6(\eta)$ is also a constant independent of coloured parameter. We let $m_5(\eta) = \beta$.

If $a_5(u,\xi,\eta) = -a_6(u,\xi,\eta)$ we should have $m_6(\eta) = 0$ and (2.21) still holds. Then $\alpha = 0$. It is clearly impossible thanks to proposition 2.2.

Combining (2.18*a*), (2.19) and (2.21) with the third equation in (1.14*a*) we obtain

$$(a_5'(u,\xi,\eta))^2 = \beta^2 - (\beta^2 - m_1(\eta)^2 + \alpha^2)a_5(u,\xi,\eta)^2 + \alpha^2 a_5(u,\xi,\eta)^4. \tag{2.23}$$

Using the symmetric operation we can show $m_1(\eta)$ is also a constant independent of coloured parameter. Let $m_1(\eta) = \gamma$. Similarly,

$$(a_1'(u,\xi,\eta))^2 = \beta^2 - (\beta^2 - \gamma^2 + \alpha^2)a_1(u,\xi,\eta)^2 + \alpha^2 a_1(u,\xi,\eta)^4 \tag{2.24}$$

holds if we combine (2.18*a*), (2.19) and (2.21) with the fourth in (1.14*a*). Substituting (2.16), (2.18*a*), (2.19) and (2.21) into the second of (2.10) we find that the algebraic curve satisfied by the weight functions $a_1(u,\xi,\eta)$ and $a_5(u,\xi,\eta)$ is

$$\alpha^2 u_1^2 u_5^2 - \beta^2 u_5^2 - \beta^2 u_1^2 + 2\beta\gamma u_1 u_5 + \beta^2 = 0. \tag{2.25}$$

From the second group of equations, i.e. (2.10) and (2.12), it is easy to obtain

$$\begin{aligned}
&1 + a_7(u,\xi,\eta)^2 - a_1(u,\xi,\eta)a_4(u,\xi,\eta) - a_5(u,\xi,\eta)a_6(u,\xi,\eta) = 0\\
&\alpha(a_1(u,\xi,\eta)a_6(u,\xi,\eta) + a_4(u,\xi,\eta)a_5(u,\xi,\eta)) = (m_5(\eta)+m_6(\eta))a_7(u,\xi,\eta)\\
&\alpha(a_1(u,\xi,\eta)^2 + a_6(u,\xi,\eta)^2 - a_4(u,\xi,\eta)^2 - a_5(u,\xi,\eta)^2) = 4m_1(\eta)a_7(u,\xi,\eta).
\end{aligned} \tag{2.26}$$

Then $m_1(\eta) + m_4(\eta) = 0$. If we perform the symmetric operation with respect to the third equation of (2.26) then we also obtain

$$\alpha(a_1(u,\xi,\eta)^2 + a_5(u,\xi,\eta)^2 - a_4(u,\xi,\eta)^2 - a_6(u,\xi,\eta)^2) = 4m_1(\xi)a_7(u,\xi,\eta). \tag{2.27}$$

Letting $\eta = \xi$ in the third equation in (2.26) and (2.27) we reduce to

$$a_6(u,\xi,\xi)^2 = a_5(u,\xi,\xi)^2. \tag{2.28}$$

Therefore, (2.16) and (2.28) give the following proposition.

Proposition 2.4. For gauge eight-vertex solutions of the coloured Yang–Baxter equation (1.1) the Hamiltonian coefficients $m_5(\eta)$ and $m_6(\eta)$ satisfy

$$m_5(\eta)^2 - m_6(\eta)^2 = 0.$$

From (2.26) and the second equation in (1.14*a*), we can calculate that the weight function $a_7(u,\xi,\eta)$ obeys

$$a_7'(u,\xi,\eta)^2 = \alpha^2 - ((m_5(\eta)+m_6(\eta))^2 - 4m_1^2(\eta) - 2\alpha^2)a_7(u,\xi,\eta)^2 + \alpha^2 a_7(u,\xi,\eta)^4. \tag{2.29}$$

Furthermore, as we said in remark 1, we can show $(m_5(\eta)^2 + m_6\eta))^2 - 2m_1(\eta)^2$ is a constant independent of coloured parameter using the symmetric operation. Let $\delta^2 = (m_5(\eta)^2 + m_6(\eta))^2 - 2m_1(\eta)^2$.

We conclude this section by the following theorem.

Theorem 2.5.* For a gauge eight-vertex-type solution, its weight functions must satisfy one of two systems of equations. The first is composed of (2.18*a*), (2.19), (2.21), (2.23), (2.24) and (2.25). The second is composed of (2.26) and (2.29).

3. Gauge eight-vertex-type solutions

In this section $\mathrm{sn}(\zeta)$ and $\mathrm{cd}(\zeta) = \mathrm{cn}(\zeta)/\,\mathrm{dn}(\zeta)$ are Jacobian elliptic functions.

Now we describe how to write down all gauge solutions of eight-vertex type of the coloured Yang–Baxter (1.1) and classify them into two types called Baxter type and free-fermion type.

3.1. Baxter-type solutions

We consider the first case in theorem 2.6. Since the curve (2.25) only includes two weight functions $a_1(u,\xi,\eta)$ and $a_5(u,\xi,\eta)$, we can parameterize $a_1(u,\xi,\eta)$ and $a_5(u,\xi,\eta)$ as one-parameter functions. If $\beta\pm\alpha\pm\gamma\neq 0$, (2.23) and (2.24) are elliptic differential equations. Therefore, the solutions should be

$$\begin{aligned}
a_1(u,\xi,\eta) &= a_4(u,\xi,\eta) = \frac{\mathrm{sn}(\lambda u + F(\xi) - F(\eta) + \mu)}{\mathrm{sn}\,(\mu)}\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1\\
a_5(u,\xi,\eta) &= a_6(u,\xi,\eta) = \pm\frac{\mathrm{sn}(\lambda u + F(\xi) - F(\eta))}{\mathrm{sn}\,(\mu)}\\
a_7(u,\xi,\eta) &= \pm k\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta))\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta) + \mu)
\end{aligned}\tag{3.1}$$

where k, as the module of Jacobi elliptic function, is an arbitrary constant.

If $\beta\pm\alpha\pm\gamma = 0$, the elliptic solutions (3.1) will degenerate into trigonometric solutions

$$\begin{aligned}
a_1(u,\xi,\eta) &= a_4(u,\xi,\eta) = \frac{\tan(\lambda u + F(\xi) - F(\eta) + \mu)}{\tan(\mu)}\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1\\
a_5(u,\xi,\eta) &= a_6(u,\xi,\eta) = \pm\frac{\tan(\lambda u + F(\xi) - F(\eta))}{\tan(\mu)}\\
a_7(u,\xi,\eta) &= \pm\tan(\lambda u + F(\xi) - F(\eta))\tan(\lambda u + F(\xi) - F(\eta) + \mu).
\end{aligned}\tag{3.2}$$

In (3.1) and (3.2) $\lambda\neq 0$, $\mu\neq 0$ are two arbitrary constants and $F(\xi)$ an arbitrary function.

3.2. Free-fermion type solutions

Now we consider the second case in theorem 2.6. According to proposition 2.4 it is divided into two subcases, $m_5(\xi) = m_6(\xi)$ and $m_5(\xi) = -m_6(\xi)$.

3.2.1. For the subcase of $m_5(\xi) = m_6(\xi)$. Let $m_5(\xi) \not\equiv 0$ (we will put the case of $m_5(\eta) = 0$ into the second subcase). It is clear that $\alpha \neq 0$ due to the second equation of (2.26). For brevity we let $\alpha = 1$ up to the solution transformation $\boldsymbol{D}$. When

$$m_5(\xi)^2 - m_1(\xi)^2 \not\equiv 0$$

and

$$m_5(\xi)^2 - m_1(\xi)^2 \not\equiv 1$$

equation (2.29) is then an elliptic differential equation and, from remarks 2 and 3, and the initial condition (1.11), should have solutions

$$a_7(u, \xi, \eta) = k\, \mathrm{sn}(\lambda u + F(\xi) - F(\eta))\, \mathrm{cd}(\lambda u + F(\xi) - F(\eta)) \tag{3.3}$$

where k, as the module of the elliptic function, and λ are two arbitrary constants and $F(\xi)$ is an arbitrary function with the constriction $k\lambda = 1$.

Substituting (3.3) into the second equation of (1.14a) and its counterpart as well as the first and second equation of (2.26) and using elliptic function identities we have

$$\mathrm{cd}^2 - \mathrm{sn}^2 + 2m_1(\eta)k\, \mathrm{cd}\,\mathrm{sn} + u_5^2 - u_1^2 = 0 \tag{3.4a}$$

$$\mathrm{cd}^2 - \mathrm{sn}^2 - 2m_1(\eta)k\, \mathrm{cd}\,\mathrm{sn} + u_6^2 - u_4^2 = 0 \tag{3.4b}$$

$$u_1u_4 + u_5u_6 - \mathrm{cd}^2 - \mathrm{sn}^2 = 0 \tag{3.4c}$$

$$u_1u_6 + u_4u_5 - 2m_5(\eta)k\, \mathrm{cd}\,\mathrm{sn} = 0 \tag{3.4d}$$

where $m_5(\eta)$ and $m_1(\eta)$ are arbitrary functions satisfying

$$m_5(\eta)^2 - m_1(\eta)^2 = \frac{1}{k^2}.$$

In the formulae (3.4) and in what follows we simply write sn and cd instead of elliptic functions $\mathrm{sn}(\lambda u + F(\xi) - F(\eta))$ and $\mathrm{cd}(\lambda u + F(\xi) - F(\eta))$ for brevity. Using the symmetric operation we also have

$$\mathrm{cd}^2 - \mathrm{sn}^2 + 2m_4(\xi)k\, \mathrm{cd}\,\mathrm{sn} + u_5^2 - u_4^2 = 0.$$

Since $m_1(\xi) + m_4(\xi) = 0$ one obtains

$$\mathrm{cd}^2 - \mathrm{sn}^2 - 2m_1(\xi)k\, \mathrm{cd}\,\mathrm{sn} + u_5^2 - u_4^2 = 0. \tag{3.4e}$$

From (3.4c), (3.4d) and (3.4a) one also obtains

$$-(\mathrm{cd}^2 + \mathrm{sn}^2)u_1 + (\mathrm{cd}^2 - \mathrm{sn}^2 + 2m_1(\eta)k\, \mathrm{sn}\,\mathrm{cd})u_4 + 2m_5(\eta)k\, \mathrm{sn}\,\mathrm{cd}\, u_5 = 0. \tag{3.5a}$$

If we do the symmetric operation with respect to the counterpart of (3.5a) then

$$-(\mathrm{cd}^2 + \mathrm{sn}^2)u_4 + (\mathrm{cd}^2 - \mathrm{sn}^2 - 2m_1(\xi)k\, \mathrm{sn}\,\mathrm{cd})u_1 + 2m_5(\xi)k\, \mathrm{sn}\,\mathrm{cd}\, u_5 = 0. \tag{3.5b}$$

Similarly, one has

$$(\mathrm{sn}^2 + 2m_1(\eta)k\, \mathrm{sn}\,\mathrm{cd} - \mathrm{cd}^2)u_5 - (\mathrm{cd}^2 + \mathrm{sn}^2)u_6 + 2m_5(\eta)k\, \mathrm{sn}\,\mathrm{cd}\, u_4 = 0 \tag{3.6a}$$

$$(\mathrm{sn}^2 + 2m_1(\xi)k\, \mathrm{cd}\,\mathrm{sn} - \mathrm{cd}^2)u_6 + (\mathrm{cd}^2 + \mathrm{sn}^2)u_5 + 2m_5(\xi)k\, \mathrm{cd}\,\mathrm{sn}\, u_4 = 0. \tag{3.6b}$$

Solving the equations (3.5a), (3.5b), (3.6a) and (3.6b) with respect to $\{u_1, u_4, u_5, u_6\}$ we have

$$\frac{a_4(u,\xi,\eta)}{a_1(u,\xi,\eta)} = \frac{H_4}{H_1} \qquad \frac{a_6(u,\xi,\eta)}{a_5(u,\xi,\eta)} = \frac{H_6}{H_5}$$

where

$$\begin{aligned} H_1 &= (m_5(\xi) + m_5(\eta))\,\mathrm{cd} + (m_1(\xi)m_5(\eta) + m_5(\xi)m_1(\eta))k\,\mathrm{sn} \\ H_4 &= (m_5(\xi) + m_5(\eta))\,\mathrm{cd} - (m_1(\xi)m_5(\eta) + m_5(\xi)m_1(\eta))k\,\mathrm{sn} \\ H_5 &= (m_5(\xi) + m_5(\eta))\,\mathrm{sn} + (m_1(\xi)m_5(\eta) - m_5(\xi)m_1(\eta))k\,\mathrm{cd} \\ H_6 &= (m_5(\xi) + m_5(\eta))\,\mathrm{sn} - (m_1(\xi)m_5(\eta) - m_5(\xi)m_1(\eta))k\,\mathrm{cd}\,. \end{aligned} \tag{3.6}$$

Let $u_1 = H_1X$, $u_4 = H_4X$ and $u_5 = H_5Y$, $u_6 = H_6Y$. From (3.4a) and (3.4e) one obtains

$$\begin{aligned} (H_1^2 - H_4^2)X^2 &= 4(m_5(\xi) + m_5(\eta))(m_5(\eta)m_1(\xi) + m_5(\xi)m_1(\eta))k\,\mathrm{sn}\,\mathrm{cd}\,X^2 \\ &= 2(m_1(\xi) + m_1(\eta)k\,\mathrm{sn}\,\mathrm{cd} \\ (H_5^2 - H_6^2)Y^2 &= 4(m_5(\xi) + m_5(\eta))(m_5(\eta)m_1(\xi) - m_5(\xi)m_1(\eta))k\,\mathrm{sn}\,\mathrm{cd}\,Y^2 \\ &= 2(m_1(\xi) - m_1(\eta))k\,\mathrm{sn}\,\mathrm{cd}\,. \end{aligned} \tag{3.7}$$

It is easy to show using $m_5(\xi)^2 - m_1(\xi)^2 = 1/k^2$ that

$$\begin{aligned} X^2 &= \frac{m_1(\xi) + m_1(\eta)}{2(m_5(\xi) + m_5(\eta))(m_1(\xi)m_5(\eta) + m_5(x)m_1(\eta))} \\ &= \frac{1 + k^2(m_5(\xi)m_5(\eta) - m_1(\xi)m_1(\eta))}{2(m_5(\xi) + m_5(\eta))^2} \\ &= \frac{-1 + k^2(m_5(\xi)m_5(\eta) + m_1(\xi)m_1(\eta))}{2(m_1(\xi)m_5(\eta) + m_5(\xi)m_1(\eta))^2k^2} \\ Y^2 &= \frac{m_1(\xi) - m_1(\eta)}{2(m_5(\xi) + m_5(\eta))(m_1(\xi)m_5(\eta) - m_5(x)m_1(\eta))} \\ &= \frac{1 + k^2(m_5(\xi)m_5(\eta) + m_1(\xi)m_1(\eta))}{2(m_5(\xi) + m_5(\eta))^2} \\ &= \frac{-1 + k^2(m_5(\xi)m_5(\eta) - m_1(\xi)m_1(\eta))}{2(m_1(\xi)m_5(\eta) - m_5(\xi)m_1(\eta))^2k^2}\,. \end{aligned}$$

Hence gauge solutions of the coloured Yang–Baxter equation (1.1) should obey the following forms,

$$\begin{aligned} a_1(u,\xi,\eta) &= A(\xi,\eta)\,\mathrm{cd}(\lambda u + F(\xi) - F(\eta)) + B(\xi,\eta)\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta)) \\ a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1 \\ a_4(u,\xi,\eta) &= A(\xi,\eta)\,\mathrm{cd}(\lambda u + F(\xi) - F(\eta)) - B(\xi,\eta)\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta)) \\ a_5(u,\xi,\eta) &= C(\xi,\eta)\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta)) + D(\xi,\eta)\,\mathrm{cd}(\lambda u + F(\xi) - F(\eta)) \\ a_6(u,\xi,\eta) &= C(\xi,\eta)\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta)) - D(\xi,\eta)\,\mathrm{cd}(\lambda u + F(\xi) - F(\eta)) \\ a_7(u,\xi,\eta) &= \pm k\,\mathrm{sn}(\lambda u + F(\xi) - F(\eta))\,\mathrm{cd}(\lambda u + F(\xi) - F(\eta)) \end{aligned} \tag{3.8}$$

where k, as the module of the elliptic functions, is an arbitrary constant and

$$\begin{aligned} A(\xi,\eta) &= \sqrt{(1 + G(\xi)G(\eta) - H(\xi)H(\eta))/2} \\ B(\xi,\eta) &= \sqrt{(-1 + G(\xi)G(\eta) + H(\xi)H(\eta))/2} \\ C(\xi,\eta) &= \delta\sqrt{(1 + G(\xi)G(\eta) + H(\xi)H(\eta))/2} \\ D(\xi,\eta) &= \delta\sqrt{(-1 + G(\xi)G(\eta) - H(\xi)H(\eta))/2}M \end{aligned} \tag{3.9}$$

where $\delta^2 = 1$ and

$$M = \frac{H(\xi)G(\eta) - G(\xi)H(\eta)}{\sqrt{(H(\xi)G(\eta) - G(\xi)H(\eta))^2}}$$

and $G(\xi)$, $H(\xi)$ satisfy $G(\xi)^2 - H(\xi)^2 = 1$. If we consider the solution transformation $\boldsymbol{D}$ then the restrictive condition $k\lambda = 1$ can be cancelled, namely, $\lambda \neq 0$ is also an arbitrary constant.

When the module $k = 1$ the Jacobian elliptic functions cd and sn should degenerate into 1 and tanh. Hence, we have

$$\begin{aligned}
a_1(u,\xi,\eta) &= A(\xi,\eta) + B(\xi,\eta)\tanh(\lambda u + F(\xi) - F(\eta)) \\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1 \\
a_4(u,\xi,\eta) &= A(\xi,\eta) - B(\xi,\eta)\tanh(\lambda u + F(\xi) - F(\eta)) \\
a_5(u,\xi,\eta) &= C(\xi,\eta)\tanh(\lambda u + F(\xi) - F(\eta)) + D(\xi,\eta) \\
a_6(u,\xi,\eta) &= C(\xi,\eta)\tanh(\lambda u + F(\xi) - F(\eta)) - D(\xi,\eta) \\
a_7(u,\xi,\eta) &= \pm\tanh(\lambda u + F(\xi) - F(\eta))
\end{aligned} \tag{3.10}$$

where $A(\xi,\eta)$, $B(\xi,\eta)$, $C(\xi,\eta)$ and $D(\xi,\eta)$ are defined by (3.9) and $\lambda \neq 0$ is an arbitrary constant.

If $m_5(\eta)^2 - m_1(\eta)^2 = 0$ then the differential equation (2.29) can be rewritten as

$$a_7'(u,\xi,\eta)^2 = \alpha^2(1 + 2a_7(u,\xi,\eta)^2 + a_7(u,\xi,\eta)^4).$$

So, following the calculation of (3.8) we can show that the gauge solutions are

$$\begin{aligned}
a_1(u,\xi,\eta) &= X\left(\frac{G(\xi)+G(\eta)}{\cos(\lambda u + F(\xi) - F(\eta))} + 2G(\xi)G(\eta)\sin(\lambda u + F(\xi) - F(\eta))\right) \\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1 \\
a_4(u,\xi,\eta) &= X\left(\frac{G(\xi)+G(\eta)}{\cos(\lambda u + F(\xi) - F(\eta))} - 2G(\xi)G(\eta)\sin(\lambda u + F(\xi) - F(\eta))\right) \\
a_5(u,\xi,\eta) &= Y\left(\frac{G(\xi)-G(\eta)}{\cos(\lambda u + F(\xi) - F(\eta))} + 2G(\xi)G(\eta)\sin(\lambda u + F(\xi) - F(\eta))\right) \\
a_6(u,\xi,\eta) &= Y\left(-\frac{G(\xi)-G(\eta)}{\cos(\lambda u + F(\xi) - F(\eta))} + 2G(\xi)G(\eta)\sin(\lambda u + F(\xi) - F(\eta))\right) \\
a_7(u,\xi,\eta) &= \pm\tan(\lambda u + F(\xi) - F(\eta))
\end{aligned} \tag{3.11}$$

where

$$X = \frac{1}{2\sqrt{G(\xi)G(\eta)}} \qquad Y = \pm X \tag{3.12}$$

and $G(\xi)$ is an arbitrary function.

3.2.2. For the subcase of $m_5(\xi) = -m_6(\xi)$. In the case of $m_5(\eta) = -m_6(\eta)$ weight functions of gauge solutions are

$$\begin{aligned}
a_1(u,\xi,\eta) &= a_4(u,\xi,\eta) = \frac{\cosh(\lambda u + F(\xi) - F(\eta))}{\cos(\mu u + G(\xi) - G(\eta))} \\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = 1 \\
a_5(u,\xi,\eta) &= -a_6(u,\xi,\eta) = \pm\frac{\sinh(\lambda u + F(\xi) - F(\eta))}{\cos(\mu u + G(\xi) - G(\eta))} \\
a_7(u,\xi,\eta) &= \pm\tan(\mu u + G(\xi) - G(\eta))
\end{aligned} \tag{3.13}$$

where λ and μ are two arbitrary constants, but not zero simultaneously, and $F(\xi)$, $G(\xi)$ are two arbitrary functions.

To prove it we first consider the subcase of $\alpha \neq 0$. Then it follows that

$$a_1(u,\xi,\eta)a_6(u,\xi,\eta) + a_4(u,\xi,\eta)a_5(u,\xi,\eta) = 0$$

by (2.26) and

$$a_1(u,\xi,\eta)a_5(u,\xi,\eta) + a_4(u,\xi,\eta)a_6(u,\xi,\eta) = 0$$

by the symmetric operation. Since $a_1(u,\xi,\eta) \neq -a_4(u,\xi,\eta)$ owing to the initial condition (1.11), so one can get

$$\begin{aligned} &a_1(u,\xi,\eta) = a_4(u,\xi,\eta) \\ &a_5(u,\xi,\eta) = -a_6(u,\xi,\eta) \\ &a_1(u,\xi,\eta)^2 - a_5(u,\xi,\eta)^2 = 1 + a_7(u,\xi,\eta)^2 \end{aligned} \tag{3.14}$$

and then $m_1(\eta) = 0$ because of the first equation of (3.14) and the condition $m_1(\xi)+m_4(\xi) = 0$. Then (2.29) has the solution

$$a_7(u,\xi,\eta) = \tan(\mu u + G(\xi) - G(\eta))$$

and hence (3.13) is true.

Second we consider the subcase of $\alpha = 0$. Then it follows from the argument of proposition 2.2 in section 2 that the weight functions are

$$\begin{aligned} &a_1(u,\xi,\eta) = A_1(\xi,\eta)\cosh u - A_6(\xi,\eta)\sinh u \\ &a_4(u,\xi,\eta) = A_4(\xi,\eta)\cosh u + A_5(\xi,\eta)\sinh u \\ &a_5(u,\xi,\eta) = A_5(\xi,\eta)\cosh u + A_4(\xi,\eta)\sinh u \\ &a_6(u,\xi,\eta) = A_6(\xi,\eta)\cosh u - A_1(\xi,\eta)\sinh u \\ &a_7(u,\xi,\eta) = A_7(\xi,\eta) \end{aligned} \tag{3.15}$$

where $A_i(\xi,\eta) = a_i(0,\xi,\eta)$ $(i = 1, 4\text{–}7)$ are some functions with respect to coloured parameters ξ, η to be determinate. It is clear that, $A_i(\xi,\eta)$ $(i = 1, 4\text{–}7)$ satisfy the pure coloured Yang–Baxter (1.2).

If we substitute the initial condition (1.11) into the ones obtained by putting $v = -u$ and $\lambda = \eta$ in the system of (1.10) and combine the resulting equation with the free-fermion condition (2.12), one can show

$$\begin{aligned} &a_5(u,\xi,\eta) = -a_5(-u,\eta,\xi) \qquad a_6(u,\xi,\eta) = -a_6(-u,\eta,\xi) \\ &a_7(u,\xi,\eta) = -a_7(-u,\eta,\xi) \qquad a_4(u,\xi,\eta) = a_1(-u,\eta,\xi). \end{aligned} \tag{3.16}$$

It is easy by (3.16) to show that

$$A_4(\xi,\eta) = A_1(\xi,\eta) \qquad A_6(\xi,\eta) = -A_5(\xi,\eta). \tag{3.17}$$

As mentioned in remark 2, all formulae obtained in this section and sections 1 and 2 should hold for the pure coloured Yang–Baxter equation (1.2) except those obtained by using the symmetric operation. So we still should have

$$\begin{aligned} &1 + A_7(\xi,\eta)^2 - A_1(\xi,\eta)A_4(\xi,\eta) - A_5(\xi,\eta)A_6(\xi,\eta) = 0 \\ &l_7(\eta)(A_1(\xi,\eta)A_6(\xi,\eta) + A_4(\xi,\eta)A_5(\xi,\eta)) = (l_5(\eta) + l_6(\eta))A_7(\xi,\eta) \\ &l_7(\eta)(A_1(\xi,\eta)^2 + A_6(\xi,\eta)^2 - A_4(\xi,\eta)^2 - A_5(\xi,\eta)^2 = 4l_1(\eta)A_7(\xi,\eta) \end{aligned} \tag{3.18}$$

where, as mentioned in remark 2, $l_i(\eta)$ $(i = 1, 4\text{–}7)$ means $(\partial/\partial\eta)A_i(\xi,\eta)|_{\xi=\eta}$. Substituting (3.17) into (3.18) we see

$$l_5(\eta) + l_6(\eta) = 0 \qquad l_1(\eta) = 0.$$

Therefore, as we did for getting (2.29) we also have

$$\left(\frac{\partial}{\partial\eta}A_7(\xi,\eta)\right)^2 = l_7(\eta)(1+2a_7(u,\xi,\eta)^2 + a_7(u,\xi,\eta)^4) \tag{3.19}$$

and then the solution for $A_i(\xi,\eta)$ is

$$\begin{aligned} A_1(\xi,\eta) &= A_4(\xi,\eta) = \frac{\cosh(F(\xi)-F(\eta))}{\cos(G(\xi)-G(\eta))} \\ A_5(\xi,\eta) &= -A_6(\xi,\eta) = \frac{\sinh(F(\xi)-F(\eta))}{\cos(G(\xi)-G(\eta))} \\ A_7(\xi,\eta) &= \tan(G(\xi)-G(\eta)) \end{aligned} \tag{3.20}$$

where F and G are two arbitrary functions of a single variable. Substituting (3.20) into (3.15) one can say (3.13) is also true for the case of $\alpha = 0$, only $\mu = 0$.

The description above tells us that, for the case of Hamiltonian coefficients $m_5(\eta) = -m_6(\eta)$, the weight functions of a gauge eight-vertex-type solution must be (3.13).

Finally, straightforward calculation and computer symbolic computation can verify the following theorem.

Theorem 3.1. Gauge eight-vertex solutions of the coloured Yang–Baxter equation (1.1) are composed of (3.1), (3.8) and (3.13) and their degenerate forms (3.2), (3.10) and (3.11).

The solutions (3.1) and (3.2) are called Baxter-type solutions. They are just the solutions for the 'zero field' eight-vertex model by Baxter [3]. The solutions (3.8), (3.13) and their degenerate forms (3.10), (3.11) satisfy the free-fermion condition and are called free-fermion-type solutions.

If we take $\lambda = 1$, $G(\xi) = \cosh(2\xi)$, $H(\xi) = \sinh(2\xi)$ and $F(\xi) = 0$ in solution (3.8) then (3.8) will reduce to the following solution

$$\begin{aligned} a_1(u,\xi,\eta) &= \cosh(\xi-\eta)\,\mathrm{cd}(u) + \sinh(\xi+\eta)\,\mathrm{sn}(u) \\ a_4(u,\xi,\eta) &= \cosh(\xi-\eta)\,\mathrm{cd}(u) - \sinh(\xi+\eta)\,\mathrm{sn}(u) \\ a_5(u,\xi,\eta) &= \cosh(\xi+\eta)\,\mathrm{sn}(u) - \sinh(\xi-\eta)\,\mathrm{cd}(u) \\ a_6(u,\xi,\eta) &= \cosh(\xi+\eta)\,\mathrm{sn}(u) + \sinh(\xi-\eta)\,\mathrm{cd}(u) \\ a_7(u,\xi,\eta) &= k\,\mathrm{sn}(u)\,\mathrm{cd}(u) \end{aligned}$$

which is given in [20].

4. General solutions

In this paper we have shown and classified all gauge eight-vertex solutions of the coloured Yang–Baxter equation (1.1). These gauge solutions and trivial solutions (1.12*a*) and (1.12*b*), together with five solution transformations discussed in section 1 will give all eight-vertex-type solutions.

If we take in (3.8) and (3.9)

$$G(\xi) = \frac{1}{\mathrm{sn}(\xi)} \qquad H(\xi) = \frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)} \qquad F(\xi) = 0 \qquad \lambda = \frac{1}{2}$$

and the solution transformation $\boldsymbol{B}$ with

$$g(u,\xi,\eta) = \sqrt{\mathrm{e}(\xi)\,\mathrm{e}(\eta)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\eta)}\,\frac{(1-\mathrm{e}(u))}{\mathrm{sn}(u/2)}$$

where the elliptic exponential

$$\mathrm{e}(\zeta) = \mathrm{cn}(\zeta) + \mathrm{i}\,\mathrm{sn}(\zeta)$$

then using addition theorems for elliptic functions $\mathrm{sn}(\zeta)$, $\mathrm{cn}(\zeta)$ and $\mathrm{dn}(\zeta)$ we can obtain the following solution given in [12]

$$\begin{aligned}
a_1(u,\xi,\eta) &= 1 - \mathrm{e}(u)\,\mathrm{e}(\xi)\,\mathrm{e}(\eta)\\
a_2(u,\xi,\eta) &= a_3(u,\xi,\eta) = \sqrt{\mathrm{e}(\xi)\,\mathrm{e}(\eta)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\eta)}\,\frac{(1-\mathrm{e}(u))}{\mathrm{sn}(u/2)}\\
a_4(u,\xi,\eta) &= \mathrm{e}(u) - \mathrm{e}(\xi)\,\mathrm{e}(\eta)\\
a_5(u,\xi,\eta) &= \mathrm{e}(\xi) - \mathrm{e}(u)\,\mathrm{e}(\eta)\\
a_6(u,\xi,\eta) &= \mathrm{e}(\eta) - \mathrm{e}(u)\,\mathrm{e}(\xi)\\
a_7(u,\xi,\eta) &= a_8(u,\xi,\eta) = -\mathrm{i}k\sqrt{\mathrm{e}(\xi)\,\mathrm{e}(\eta)\,\mathrm{sn}(\xi)\,\mathrm{sn}(\eta)}(1-\mathrm{e}(u))\,\mathrm{sn}(u/2)
\end{aligned}$$

the detailed calculations being omitted.

Similarly, all non-trivial general solutions can also be classified into two types. The first are Baxter-type solutions if they can be obtained via gauge Baxter solutions and some solution transformations. The second are free-fermion solutions if they can be obtained via gauge free-fermion solutions and some solution transformations.

According to the standard method by Baxter, for a given R-matrix the spin-chain Hamiltonian is generally of the following form,

$$H = \sum_{j=1}^{N}(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z + \tfrac{1}{2}h(\sigma_j^z + \sigma_{j+1}^z))$$

where σ^x, σ^y and σ^z are Pauli matrices and the coupling constants are

$$J_x = \tfrac{1}{4}(m_5+m_6+m_7+m_8) \qquad J_y = \tfrac{1}{4}(m_5+m_6-m_7-m_8)$$
$$J_z = \tfrac{1}{4}(m_1-m_3+m_4-m_2) \qquad h = \tfrac{1}{4}(m_1-m_3-m_4+m_2).$$

In this paper we have proved that the Hamiltonian coefficients of a gauge solution must obey

$$m_1^2 = m_4^2 \qquad m_5^2 = m_6^2.$$

It follows from the solution transformations $\boldsymbol{B}$ and $\boldsymbol{D}$ that

$$(m_1-m_3)^2 = (m_4-m_2)^2 \qquad m_5^2 = m_6^2$$

for general solutions. This clearly describes the relation between classifications of eight-vertex-type solutions and spin-chain Hamiltonians. For example, if $J_x + J_y = h$, $J_z = 0$, i.e. a special free-fermion model in a magnetic field, then one has $m_5 = m_1 - m_3 = -m_4 + m_2$. The corresponding solution of the coloured Yang–Baxter equation should be (3.11). Then the transfer matrix is of trigonometric function type.

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